

# Existence of a Radner equilibrium in a model with transaction costs

Kim Weston<sup>1</sup>

The University of Texas at Austin

Department of Mathematics

Austin, TX 78712, USA

February 7, 2017

## Abstract

We prove the existence of a Radner equilibrium in a model with proportional transaction costs on an infinite time horizon. Two agents receive exogenous, unspanned income and choose between consumption and investing into an annuity. After establishing the existence of a discrete-time equilibrium, we show that the discrete-time equilibrium converges to a continuous-time equilibrium model. The continuous-time equilibrium provides an explicit formula for the equilibrium interest rate in terms of the transaction cost parameter. We show analytically that the interest rate can be either increasing or decreasing in the transaction costs depending on the agents' risk parameters.

*Keywords:* Transaction costs, Radner equilibrium, Shadow prices, Incompleteness

*JEL Classification:* D52, G12, G11

*Mathematics Subject Classification (2010):* 91B51, 91B25

## 1 Introduction

We study an incomplete Radner equilibrium with proportional transaction costs. Two exponential investors receive unspanned income, consume, and trade in an annuity market on an infinite time horizon. Transaction costs influence asset prices, yet most asset pricing models with transaction costs take asset prices as given. The individual agent's optimal investment problem in the presence of proportional transaction costs is well-studied and dates back to [15] and [7]. The current paper provides an example

---

<sup>1</sup>The author acknowledges support by the National Science Foundation under Grant No. DMS-1606253. Any opinions, findings and conclusions or recommendations expressed in this material are those of the author and do not necessarily reflect the views of the National Science Foundation (NSF).

of an equilibrium with proportional transaction costs, which derives all traded asset prices endogenously.

Proving the existence of a general equilibrium is a challenging problem on its own, and frictions exacerbate the difficulties. Existing work on equilibria with transaction costs lacks the ability to endogenously derive asset prices while providing rigorous justification in a finite-agent Radner equilibrium. The works [18], [14], and [6] rely on an exogenously specified bank account, while [14] and [1] provide numerics but no existence result. Models with a continuum of agents are introduced in [19], [18], and [8]. In contrast, we introduce transaction costs into a finite-agent Radner equilibrium with unspanned income and consumption over an infinite time horizon. This set-up is longstanding without transaction costs; see, for instance, [2] and [20]. We prove the existence of a proportional transaction cost Radner equilibrium for two exponential agents with unspanned income, where the traded security price is determined endogenously.

Single-agent models with proportional transaction costs in continuous time often lead to optimal strategies that operate on a local time time scale, which is singular with respect to the Lebesgue measure ( $dt$ ); see, for instance, [7] and [16]. This presents a challenge for continuous-time equilibrium models: When consumption occurs on a  $dt$ -time scale and transaction costs are leaving the economy on a local time scale, what is the appropriate time scale for market clearing? As the time discretization gets finer, our discrete-time equilibrium passes to a continuous-time equilibrium with investment and consumption occurring on the same  $dt$ -time scale. We show that our discrete-time equilibrium converges as the time step goes to zero, and we show that the limit is indeed a continuous-time equilibrium.

Understanding the effects of transaction costs analytically is often not possible. In [9], [17], and [12], the authors derive a Taylor expansion for the single-agent value function and no-trade boundaries for small transaction costs. In equilibrium, we seek to understand the effect of transaction costs on the equilibrium interest rate. We derive an explicit formula for the interest rate in terms of the transaction costs in continuous time.

We employ a shadow price approach to establish an equilibrium. Shadow prices represent the traded asset price in a least-favorable frictionless market completion, where the optimal investment and consumption strategies align between the frictionless shadow market and the transaction cost market. Shadow prices for proportional transaction costs were introduced by [10] and [4] and have since been established in increasingly greater generality; see, for example, [11] and [5]. Because least-favorability is investor specific, each economic agent will select her own frictionless shadow market to perform utility maximization. We link the investor-specific shadow markets using

a “closeness” condition in equilibrium. We show that a unique equilibrium asset price is only guaranteed when a trade occurs. Otherwise, agents’ shadow prices allow for a range of prices consistent with the equilibrium.

Several components of this equilibrium example are crucial for obtaining our results. We rely on the agents’ exponential preferences and income processes with independent increments for tractability of the single-agent problem similar to [20], [3], and [13]. In a frictionless model, an annuity is spanned by a bank account, and vice versa. With transaction costs, we cannot freely move between an annuity and the bank account as the traded security. We work with the annuity as the traded security similar to [8]. This choice yields trading strategies in which the agents choose to do the same thing at every time point: either buy, sell, or trade nothing. Theorem 4.2 proves that the constant interest rate equilibrium obtained by trading in the annuity is not possible when the bank account is the traded security.

The paper is organized as follows. Section 2 describes the discrete-time equilibrium and proves its existence in Theorem 2.4. Section 3 considers a continuous-time equilibrium model. The existence of an equilibrium is established in Theorem 3.4, and it is shown to be the limit of discrete-time equilibria. We also derive an explicit formula for the equilibrium interest rate in terms of the transaction cost parameter. Section 4 discusses a transaction cost equilibrium with a traded bank account. The proofs are contained in Section 5.

## 2 Discrete-Time Equilibrium

We consider a discrete-time infinite time horizon Radner equilibrium without a risky asset. There is a single consumption good, which we take to be the numeraire. Time is divided into intervals  $[t_n, t_{n+1})$ ,  $n \geq 0$ , where  $t_n := n\Delta$  and  $\Delta > 0$ . An annuity, denoted by  $A$ , is in one-net supply and is available to trade with an exogenously specified proportional transaction cost  $\lambda \in [0, 1)$ . One share in the annuity delivers consumption units at a rate of one per unit time over all future time intervals. Thus, a share in the annuity will deliver  $\Delta$  consumption units over each time interval  $[t_n, t_{n+1})$ .

The risk-free rate  $r > 0$  will be determined endogenously in equilibrium using the equilibrium annuity values. We will focus on equilibria allowing for constant, positive interest rates. The annuity dynamics are given by

$$A_{t_{n+1}} - A_{t_n} = (A_{t_n} r - 1) \Delta, \quad A_0 > 0.$$

In this case, the annuity value will be the constant  $A_{t_n} = A = 1/r$ .

Each agent has the exogenous income stream  $Y_i = (Y_{it_n})_{n \geq 0}$  given by,

$$Y_{it_{n+1}} = Y_{it_n} + \mu_i \Delta + \sqrt{\Delta} \sigma_i Z_{it_{n+1}}, \quad Y_{i0} \in \mathbb{R},$$

where  $\mu_i \in \mathbb{R}$ ,  $\sigma_i > 0$ , and  $Z_{it_{n+1}} \sim \mathcal{N}(0, 1)$  for  $i = 1, 2$ . The random variables  $(Z_{it_n})_{n \geq 1}$  are independent, and  $Z_1$  and  $Z_2$  are possibly correlated. The agents are also endowed with an initial allocation of annuity shares  $\theta_{i0} \in \mathbb{R}$  such that  $\theta_{10} + \theta_{20} = 1$ .

The flow of information in this economy is given by  $\mathbb{F} = (\mathcal{F}_{t_n})_{n \geq 0}$ , where  $\mathcal{F}_{t_n} = \sigma(Z_{it_1}, \dots, Z_{it_n} : i = 1, 2)$ . All processes are assumed to be adapted to  $\mathbb{F}$ , and all agents share the same filtration and probability  $\mathbb{P}$ . All equalities are assumed to hold  $\mathbb{P}$ -almost surely.

## 2.1 Individual Agent Problems

Rather than deal directly with an optimization problem in a market with frictions, we cast each individual investor's problem as a problem in her own frictionless shadow market. In equilibrium, the agents' shadow markets will be related, and a unique (non-shadow) equilibrium price for the traded annuity will exist when a trade occurs. Yet the individual optimization problems are treated in isolation as frictionless. Therefore, only in the next section (Section 2.2) will the parameter  $\lambda$  appear.

We first consider the single-agent investment and consumption problem for agent  $i \in \{1, 2\}$ . At time  $t_n \geq 0$ , agent  $i$  chooses to consume  $c_{t_n}$  units of the consumption good and invest  $\theta_{t_n}$  shares in the annuity beginning with an initial allocation of  $\theta_0$ . We consider equilibria for which the value of the shadow annuity and shadow interest rate (to be determined endogenously in equilibrium) are constants  $A_{it_n} = A_i = 1/r_i$  and  $r_i > 0$ , respectively.

For a given investment strategy  $\theta$ , agent  $i$ 's shadow wealth is defined by

$$X_{it_n} := \theta_{t_n} A_{it_n},$$

with the self-financing condition

$$(\theta_{t_{n+1}} - \theta_{t_n}) A_{it_{n+1}} = (Y_{it_n} - c_{t_n} + \theta_{t_n}) \Delta, \quad n \geq 0. \quad (2.1)$$

For a given consumption and investment strategy  $(c, \theta)$ , the wealth evolves as

$$X_{it_{n+1}}^c - X_{it_n}^c = (X_{it_n}^c r_i + Y_{it_n} - c_{t_n}) \Delta, \quad X_{i0} = \theta_0 A_{i0}.$$

Given a consumption strategy  $c$  and an initial share allocation  $\theta_0$ , the self-financing condition dictates the investment strategy  $\theta$ .

We consider agents with exponential preferences over running consumption; that is, agent  $i$ 's utility function is  $c \mapsto -e^{-\alpha_i c}$  for  $\alpha_i > 0$ . The agents prefer consumption now to consumption later, which is measured by their time-preference parameters  $\beta_i > 0$ .

**Definition 2.1.** A consumption strategy  $c$  is called *admissible for agent  $i$*  if  $c$  satisfies the transversality requirement

$$\mathbb{E} \left[ \exp \left( -\beta_i t_n - \alpha_i r_i X_{t_n}^c - \alpha_i Y_{t_n} \right) \right] \longrightarrow 0 \quad \text{as } n \rightarrow \infty.$$

In this case, we write  $c \in \mathcal{A}_i^\Delta$ .

The value function is defined by

$$V_i^\Delta(x, y) := \sup_{c \in \mathcal{A}_i^\Delta} \sum_{n=0}^{\infty} \mathbb{E} \left[ e^{-\beta_i t_n} U_i(c_{t_n}) \right], \quad x, y \in \mathbb{R}. \quad (2.2)$$

**Theorem 2.2.** Agent  $i$ 's optimal consumption and wealth process in (2.2) are given by

$$\hat{c}_{it_n} = r_i \hat{X}_{it_n} + Y_{it_n} + \frac{1}{\alpha_i r_i \Delta} \left( \tilde{\beta}_i \Delta - \log(1 + r_i \Delta) \right) \quad (2.3)$$

and

$$\hat{X}_{it_n} = \frac{\theta_{i0}}{r_i} + \frac{t_n}{\alpha_i r_i} \left( \frac{1}{\Delta} \log(1 + r_i \Delta) - \tilde{\beta}_i \right), \quad (2.4)$$

where  $\tilde{\beta}_i := \beta_i + \alpha_i \mu_i - \frac{\alpha_i^2 \sigma_i^2}{2}$ . Moreover, the value function can be expressed in the form

$$V_i^\Delta(x, y) = J_i^\Delta(x, y) := -\frac{1}{r_i \Delta} (1 + r_i \Delta)^{1 + \frac{1}{r_i \Delta}} \exp \left( -\alpha_i r_i x - \alpha_i y - \frac{\tilde{\beta}_i}{r_i} \right).$$

## 2.2 Equilibrium

The definition of equilibrium must allow us to relate both agents' willingness to trade, even when no trade occurs due to frictions. Shadow prices provide us with this mechanism and a way to compute the range of annuity values consistent with an equilibrium. Typically, shadow prices are used as a tool to establish properties of an original model with frictions; see, for example, [11] and [5]. Here, we work with shadow prices directly, and we subsequently determine transaction cost models consistent with our agents' shadow markets.

**Definition 2.3.** For the transaction cost parameter  $\lambda \in [0, 1)$ , an *equilibrium with transaction costs* is given by a collection of processes  $(A_i, \hat{c}_i, \hat{\theta}_i)_{i=1,2}$  such that

(i) Real and financial markets clear for each  $n \geq 0$ :

$$\sum_{i=1}^2 \hat{c}_{it_n} \Delta = \Delta + \sum_{i=1}^2 Y_{it_n} \Delta - 2\lambda \left| \hat{\theta}_{1t_{n+1}} - \hat{\theta}_{1t_n} \right| A_{t_{n+1}} \quad \text{and} \quad \hat{\theta}_{1t_n} + \hat{\theta}_{2t_n} = 1,$$

where in the event of a trade, we define  $A_{t_{n+1}} := \frac{A_{it_{n+1}}}{1+\lambda}$  if agent  $i \in \{1, 2\}$  purchases a positive number of annuity shares; that is,  $\theta_{it_{n+1}} - \theta_{it_n} > 0$ .

(ii) For each agent  $i = 1, 2$ , the consumption and investment strategies,  $\hat{c}_i$  and  $\hat{\theta}_i$  with  $\hat{\theta}_{i0} = \theta_{i0}$ , are optimal with the shadow annuity price  $A_i$ :

$$V_i^\Delta(\theta_{i0} A_{i0}) = - \sum_{n=0}^{\infty} \mathbb{E} \left[ e^{-\beta_i t_n} e^{-\alpha_i \hat{c}_{it_n}} \right].$$

(iii) The shadow markets remain “close enough” to the underlying transaction cost market in the following sense: For each  $n \geq 0$ ,

$$\frac{A_{1t_n}}{A_{2t_n}} \in \left[ \frac{1-\lambda}{1+\lambda}, \frac{1+\lambda}{1-\lambda} \right].$$

Moreover, for  $n \geq 1$ , if  $\hat{\theta}_{1t_n} - \hat{\theta}_{1t_{n-1}} > 0$  then  $A_{1t_n} = A_{2t_n} \cdot \frac{1+\lambda}{1-\lambda}$ . If  $\hat{\theta}_{1t_n} - \hat{\theta}_{1t_{n-1}} < 0$  then  $A_{1t_n} = A_{2t_n} \cdot \frac{1-\lambda}{1+\lambda}$ .

*Remark 2.1.* Let us consider a single-agent optimization problem for a risky asset with frictions  $S$  and a shadow price  $\tilde{S}$ . The shadow price along with the optimal trading strategy  $\hat{\theta}$  will satisfy  $\tilde{S}_{t_n} \in [(1-\lambda)S_{t_n}, (1+\lambda)S_{t_n}]$ ,  $\tilde{S}_{t_n} = (1-\lambda)S_{t_n}$  when  $\hat{\theta}_{t_n} - \hat{\theta}_{t_{n-1}} < 0$ , and  $\tilde{S}_{t_n} = (1+\lambda)S_{t_n}$  when  $\hat{\theta}_{t_n} - \hat{\theta}_{t_{n-1}} > 0$ . Condition (iii) in Definition 2.3 enforces this relationship between both agents’ shadow markets and the underlying market.

Since each agent optimizes in her own shadow market while maintaining the “closeness” condition (iii), a unique market annuity rate is only guaranteed when trade occurs. When trade does not occur in a given period, there is a range of possible annuity values (and corresponding interest rates) consistent with equilibrium.

The following is the main result of the section. The proof is in Section 5.

**Theorem 2.4.** *Let  $\tilde{\beta}_i := \beta_i + \alpha_i \mu_i - \frac{\alpha_i^2 \sigma_i^2}{2}$ , and assume that  $\tilde{\beta}_i$  is strictly positive for  $i = 1, 2$ . For  $\lambda \in [0, 1)$ , there exists an equilibrium with strictly positive constant shadow interest rates  $r_1, r_2$  and constant shadow annuity values  $A_1 = 1/r_1$ ,  $A_2 = 1/r_2$ . The optimal consumption and wealth processes for investor  $i = 1, 2$ , are given by (2.3) and (2.4), respectively.*

**Case 1:** *A no-trade equilibrium occurs if*

$$\frac{e^{\tilde{\beta}_2 \Delta} - 1}{e^{\tilde{\beta}_1 \Delta} - 1} \in \left[ \frac{1-\lambda}{1+\lambda}, \frac{1+\lambda}{1-\lambda} \right]. \quad (2.5)$$

In this case,

$$r_1 = \frac{e^{\tilde{\beta}_1 \Delta} - 1}{\Delta} \quad \text{and} \quad r_2 = \frac{e^{\tilde{\beta}_2 \Delta} - 1}{\Delta}.$$

The range of possible constant, non-shadow interest rates that are consistent with this equilibrium is given by

$$r \in \left[ \frac{1 - \lambda}{\Delta} \left( e^{\max(\tilde{\beta}_1, \tilde{\beta}_2) \Delta} - 1 \right), \frac{1 + \lambda}{\Delta} \left( e^{\min(\tilde{\beta}_1, \tilde{\beta}_2) \Delta} - 1 \right) \right] \neq \emptyset.$$

**Case 2:** There exists an equilibrium in which agent 1 will purchase shares of the annuity in equilibrium at all times  $t_n \geq 0$  (while agent 2 sells shares) if

$$\frac{e^{\tilde{\beta}_2 \Delta} - 1}{e^{\tilde{\beta}_1 \Delta} - 1} > \frac{1 + \lambda}{1 - \lambda}, \quad (2.6)$$

where the interest rate  $r > 0$  is uniquely determined by

$$\left( 1 + \frac{r \Delta}{1 - \lambda} \right)^{\frac{1}{\alpha_2 \Delta}} \left( 1 + \frac{r \Delta}{1 + \lambda} \right)^{\frac{1}{\alpha_1 \Delta}} = e^{\frac{\tilde{\beta}_1}{\alpha_1} + \frac{\tilde{\beta}_2}{\alpha_2}}, \quad (2.7)$$

and the shadow interest rates are given in terms of

$$r = r_1(1 + \lambda) = r_2(1 - \lambda).$$

*Remark 2.2.* If the inequality (2.6) is flipped so that

$$\frac{e^{\tilde{\beta}_1 \Delta} - 1}{e^{\tilde{\beta}_2 \Delta} - 1} > \frac{1 + \lambda}{1 - \lambda},$$

then we can conclude an analogous result in which the roles of agent 1 and 2 in are interchanged.

### 3 Continuous-Time Equilibrium

Establishing the existence of an equilibrium in discrete time was the first step to determining the form of a continuous-time transaction cost equilibrium. In our simple setting, the presence of only one traded security allows for an optimal continuous-time trading strategy that is absolutely continuous with respect to the Lebesgue measure ( $dt$ ). Since consumption occurs on the  $dt$ -time scale, the absolute continuity property will allow transaction costs to be paid on the same  $dt$ -time scale in the real goods market. Typical single-agent utility maximization problems with transaction costs optimally trade a risky security on a local time time scale; see, for example, [16].

In this section, we consider the continuous-time infinite time horizon Radner equi-

librium. The only traded security is an annuity  $A$ , which is in one-net supply and available to trade with the proportional transaction cost rate  $\lambda \in [0, 1)$ . The risk-free rate  $r > 0$  will be determined endogenously in equilibrium using the equilibrium annuity values. We again focus on equilibria allowing for constant, positive interest rates, in which case, the annuity value will be the constant  $A = 1/r$ .

Each of the two agents has an exogenous income stream given by  $Y_i = (Y_{it})_t$ ,  $i = 1, 2$ , with dynamics

$$dY_{it} = \mu_i dt + \sigma_i dB_{it}, \quad Y_{i0} \in \mathbb{R},$$

where  $\mu_i \in \mathbb{R}$ ,  $\sigma_i > 0$ , and  $B_1$  and  $B_2$  are possibly correlated Brownian motions. The agents are also endowed with an initial allocation of shares in the annuity  $\theta_{i0} \in \mathbb{R}$  such that  $\theta_{10} + \theta_{20} = 1$ .

The flow of information in the economy is given by  $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ , where  $\mathcal{F}_t = \sigma(B_{1u}, B_{2u} : 0 \leq u \leq t)$ . All process are assumed to be adapted to  $\mathbb{F}$ , and all agents share the same filtration.

### 3.1 Individual Agent Problems

We consider the single agent investment and consumption problem for agent  $i$ 's shadow market,  $i \in \{1, 2\}$ . We focus on models where the value of the shadow annuity and shadow interest rate (to be determined endogenously in equilibrium) are constants  $A_{it} = A_i = 1/r_i$  and  $r_i > 0$ , respectively.

For a given investment strategy  $\theta$ , agent  $i$ 's shadow wealth is defined by  $X_{it} := \theta_t A_{it}$ . For a measurable, adapted consumption process  $c = (c_t)_t$  for which  $\int_0^T |c_t| dt < \infty$   $\mathbb{P}$ -almost surely for all  $T > 0$ , the shadow wealth process associated with  $c$  evolves like

$$dX_{it}^c = (X_{it}^c r_i - c_t + Y_{it}) dt, \quad X_{i0}^c = \theta_{i0}/r_i \in \mathbb{R}.$$

As in the discrete-time case, we consider agents with exponential preferences over running consumption with risk aversion  $\alpha_i > 0$  and time-preference parameter  $\beta_i > 0$ .

**Definition 3.1.** Let  $i \in \{1, 2\}$ . A consumption process  $c = (c_t)_t$  is called *admissible for agent  $i$*  if the transversality condition holds:

$$\lim_{t \rightarrow \infty} \mathbb{E} \left[ e^{-\beta_i t - \alpha_i r_i X_t^c - \alpha_i Y_{it}} \right] = 0.$$

In this case, we write  $c \in \mathcal{A}_i$ .



For  $i \in \{1, 2\}$ , agent  $i$ 's value function is given by

$$V_i(x, y) := \sup_{c \in \mathcal{A}_i} -\mathbb{E} \int_0^\infty e^{-\beta_i t - \alpha_i c t} dt, \quad x, y \in \mathbb{R}.$$

We show in Theorem 3.2 (below) that  $V_i = J_i$ , where

$$J_i(x, y) = -\frac{1}{r_i} \exp \left( -\alpha_i r_i x - \alpha_i y + 1 - \frac{\tilde{\beta}_i}{r_i} \right). \quad (3.1)$$

We note that  $V_i^\Delta(x, y)\Delta \rightarrow J_i(x, y)$  as  $\Delta \rightarrow 0$ , where  $V_i^\Delta$  is defined by (2.2).

The following result establishes the individual agent optimal investment strategies. The proof is omitted, as it does not vary substantially from the discrete-time case.

**Theorem 3.2.** *For  $i = 1, 2$ , let  $\tilde{\beta}_i := \beta_i + \alpha_i \mu_i - \frac{\alpha_i^2 \sigma_i^2}{2}$ . The optimal consumption policy and wealth process for agent  $i$  are given by*

$$\hat{c}_{it} = r_i \hat{X}_{it} + Y_{it} + \frac{\tilde{\beta}_i}{r_i \alpha_i} - \frac{1}{\alpha_i}, \quad (3.2)$$

$$\hat{X}_{it} = X_t^{\hat{c}_i} = \frac{\theta_{i0}}{r_i} + \frac{1}{\alpha_i} \left( 1 - \frac{\tilde{\beta}_i}{r_i} \right) t. \quad (3.3)$$

Moreover, the value function coincides with (3.1); that is,  $V_i = J_i$ .

## 3.2 Equilibrium in Continuous-Time

In addition to establishing the existence of an equilibrium, we are interested in how the equilibrium interest rate depends on  $\lambda$ .

**Definition 3.3.** For the transaction cost parameter  $\lambda \in [0, 1)$ , an *equilibrium with transaction costs* is given by a collection of processes  $(A_i, \hat{c}_i, \hat{\theta}_i)_{i=1,2}$  such that

- (i) For  $i = 1, 2$ , the optimal investment strategy  $\hat{\theta}_i$  is differentiable in time with derivative  $\hat{\theta}'_{it}$ .
- (ii) Real and financial markets clear for all  $t \geq 0$ :

$$\sum_{i=1}^2 \hat{c}_{it} = 1 + \sum_{i=1}^2 Y_{it} - 2\lambda \left| \hat{\theta}'_{1t} \right| A_t \quad \text{and} \quad \hat{\theta}_{1t} + \hat{\theta}_{2t} = 1,$$

where in the event of a trade, we define  $A_t := \frac{A_{it}}{1+\lambda}$  if agent  $i \in \{1, 2\}$  purchases a positive number of annuity shares; i.e.,  $\hat{\theta}'_{it} > 0$ .

- (iii) For each agent  $i = 1, 2$ , the consumption and investment strategies,  $\hat{c}_i$  and  $\hat{\theta}_i$  with

$\hat{\theta}_{i0} = \theta_{i0}$ , are optimal with the annuity price  $A_i$ :

$$V_i(\theta_{i0}A_{i0}) = - \int_0^\infty \mathbb{E} \left[ e^{-\beta_i t} e^{-\alpha_i \hat{c}_{it}} \right] dt.$$

(iv) The shadow markets remain “close enough” to the underlying transaction cost market in the following sense: For all  $t \geq 0$ ,

$$\frac{A_{1t}}{A_{2t}} \in \left[ \frac{1-\lambda}{1+\lambda}, \frac{1+\lambda}{1-\lambda} \right].$$

Moreover, if  $\hat{\theta}'_{1t} > 0$  then  $A_{1t} = A_{2t} \cdot \frac{1+\lambda}{1-\lambda}$ . If  $\hat{\theta}'_{1t} < 0$  then  $A_{1t} = A_{2t} \cdot \frac{1-\lambda}{1+\lambda}$ .

*Remark 3.1.* When trade occurs, we are able to define  $A_t := \frac{A_{it}}{1+\lambda}$ , where agent  $i \in \{1, 2\}$  purchases a positive number of shares,  $\hat{\theta}'_{it} > 0$ . In this case, if we consider constant interest rate equilibria, then the equilibrium interest rate is uniquely determined by  $r = 1/A = 1/A_t$ .

The following result establishes an equilibrium for the continuous-time model. The proof is omitted, as it mirrors the proof of Theorem 2.4.

**Theorem 3.4.** *Let  $\tilde{\beta}_i := \beta_i + \alpha_i \mu_i - \frac{\alpha_i^2 \sigma_i^2}{2}$ , and assume that  $\tilde{\beta}_i$  is strictly positive for  $i = 1, 2$ . For  $\lambda \in [0, 1)$ , there exists an equilibrium with strictly positive constant shadow interest rates  $r_1, r_2$  and constant shadow annuity values  $A_1 = 1/r_1$ ,  $A_2 = 1/r_2$ . The optimal consumption policies and wealth processes are given by (3.2) and (3.3), respectively.*

**Case 1:** *A no-trade equilibrium occurs if*

$$\frac{\tilde{\beta}_2}{\tilde{\beta}_1} \in \left[ \frac{1-\lambda}{1+\lambda}, \frac{1+\lambda}{1-\lambda} \right]. \quad (3.4)$$

*In this case,  $r_1 = \tilde{\beta}_1$  and  $r_2 = \tilde{\beta}_2$ . The range of possible (non-shadow) interest rates that are consistent with this equilibrium is  $r \in \left[ (1-\lambda) \max(\tilde{\beta}_1, \tilde{\beta}_2), (1+\lambda) \min(\tilde{\beta}_1, \tilde{\beta}_2) \right] \neq \emptyset$ .*

**Case 2:** *There exists an equilibrium in which agent 1 will purchase shares of the annuity in equilibrium at all times  $t \geq 0$  (while agent 2 sells shares) if*

$$\frac{\tilde{\beta}_2}{\tilde{\beta}_1} > \frac{1+\lambda}{1-\lambda}. \quad (3.5)$$

*In this case, the interest rate  $r > 0$  is determined by*

$$r = \frac{\tilde{\beta}_1/\alpha_1 + \tilde{\beta}_2/\alpha_2}{\frac{1}{\alpha_1(1+\lambda)} + \frac{1}{\alpha_2(1-\lambda)}}. \quad (3.6)$$

The shadow interest rates are given by  $r = (1 + \lambda)r_1 = (1 - \lambda)r_2$ .

Our method of solving for an equilibrium is the same for zero and non-zero transaction costs, which allows us to easily compare the endogenous variables as  $\lambda$  varies. The simplicity of the continuous-time model lends itself to further study as the transaction costs tends to zero. Depending on the agents' risk aversions, the equilibrium interest rate may be increasing or decreasing as  $\lambda$  tends to zero.

**Corollary 3.5** (The Effects of Small Transaction Costs). *Suppose that  $\tilde{\beta}_2 > \tilde{\beta}_1 > 0$ . For  $\lambda \in [0, \frac{\tilde{\beta}_2 - \tilde{\beta}_1}{\tilde{\beta}_1 + \tilde{\beta}_2})$ , the equilibrium interest rate exists, is unique among constant interest rate equilibria, and has the explicit form given by*

$$r = r(\lambda) = \frac{\tilde{\beta}_1/\alpha_1 + \tilde{\beta}_2/\alpha_2}{\frac{1}{\alpha_1(1+\lambda)} + \frac{1}{\alpha_2(1-\lambda)}}.$$

Theorem 3.6 proves that the discrete-time interest rate passes to the continuous-time equilibrium rate as the time step  $\Delta$  tends to zero.

**Theorem 3.6.** *Suppose that*

$$\frac{\tilde{\beta}_2}{\tilde{\beta}_1} > \frac{1 + \lambda}{1 - \lambda}.$$

*Let  $r(\Delta)$  be the solution to (2.7) corresponding to the time step  $\Delta > 0$ , and let  $r(0)$  be the continuous-time equilibrium interest rate given by (3.6). Then  $r(\Delta) > 0$  is the unique interest rate among constant interest rate equilibria for sufficiently small  $\Delta$ , and  $r(\Delta) \rightarrow r(0)$  as  $\Delta \rightarrow 0$ .*

*Remark 3.2.* When  $\frac{\tilde{\beta}_1}{\tilde{\beta}_2} > \frac{1+\lambda}{1-\lambda}$ , we can conclude analogous results to Theorem 3.4 Case 2 and Theorem 3.6 in which the roles of agent 1 and 2 are interchanged. An analogous result to Corollary 3.5 holds for  $\tilde{\beta}_1 > \tilde{\beta}_2 > 0$ .

## 4 The Bank Account as the Traded Security

Transaction costs in our model prevent us from trading freely between the annuity and a bank account. Using an annuity as our traded security allows for constant shadow interest rates and trading strategies that are the same at every time point: either the agents buy, sell, or trade nothing. The simple structure of transaction cost equilibria with a traded annuity is not possible when the bank account is traded instead.

In this section, we consider a discrete-time equilibrium with transaction costs when the bank account is the traded security. Theorem 4.2 proves that the traded bank account model prevents a constant-interest rate transaction cost equilibrium.

In contrast to the annuity, the bank account is a financial asset in zero-net supply. For  $i = 1, 2$ , the shadow bank account  $B_i$  has the associated interest rate process

$r_i = (r_{it_n})_{n \geq 0}$  and is given by  $B_{i0} = 1$  and

$$B_{it_n} = (1 + r_{i0}\Delta) \cdot \dots \cdot (1 + r_{it_{n-1}}\Delta), \quad n \geq 1.$$

We focus on equilibria yielding constant shadow interest rates  $r_{it_n} = r_i$ , as in the traded annuity case.

For a given investment strategy  $\theta$ , agent  $i$ 's shadow wealth is given by  $X_{it_n} := \theta_{t_n} B_{it_n}$ . Since the bank account is in zero-net supply, the self-financing condition in (2.1) will be replaced by

$$(\theta_{t_{n+1}} - \theta_{t_n}) B_{it_{n+1}} = (Y_{it_n} - c_{t_n} + \theta_{t_n}) \Delta, \quad n \geq 0.$$

Thus, for a given consumption and investment strategy  $(c, \theta)$ , the shadow wealth evolves like

$$X_{it_{n+1}}^c - X_{it_n}^c = (X_{it_n}^c r_i + Y_{it_n} - c_{t_n}) \Delta, \quad X_{i0} = \theta_0 B_{i0} = \theta_0.$$

The definitions of admissibility and the value function are unchanged from Definition 2.1 and (2.2). As such, Theorem 2.2 holds for the frictionless shadow market with a bank account carrying a constant interest rate.

**Definition 4.1.** For the transaction cost parameter  $\lambda \in [0, 1)$ , a *transaction cost equilibrium with a bank account* is given by a collection of processes  $(r_i, \hat{c}_i, \hat{\theta}_i)_{i=1,2}$  such that

- (i) Real and financial markets clear for each  $n \geq 0$ :

$$\sum_{i=1}^2 \hat{c}_{it_n} \Delta = \sum_{i=1}^2 Y_{it_n} \Delta - 2\lambda \left| \hat{\theta}_{1t_{n+1}} - \hat{\theta}_{1t_n} \right| B_{t_{n+1}} \quad \text{and} \quad \hat{\theta}_{1t_n} + \hat{\theta}_{2t_n} = 1,$$

where in the event of a trade, we define  $B_{t_{n+1}} := \frac{B_{it_{n+1}}}{1+\lambda}$  if agent  $i \in \{1, 2\}$  purchases a positive number of annuity shares; that is,  $\theta_{it_{n+1}} - \theta_{it_n} > 0$ .

- (ii) For each agent  $i = 1, 2$ , the consumption and investment strategies,  $\hat{c}_i$  and  $\hat{\theta}_i$  with  $\hat{\theta}_{i0} = \theta_{i0}$ , are optimal with the shadow bank account value  $B_i$ :

$$V_i^\Delta(\theta_{i0}) = - \sum_{n=0}^{\infty} \mathbb{E} \left[ e^{-\beta_i t_n} e^{-\alpha_i \hat{c}_{it_n}} \right].$$

- (iii) The shadow markets remain “close enough” to the underlying transaction cost market in the following sense: For each  $n \geq 1$ ,

$$\frac{B_{1t_n}}{B_{2t_n}} \in \left[ \frac{1-\lambda}{1+\lambda}, \frac{1+\lambda}{1-\lambda} \right].$$

Moreover, if  $\hat{\theta}_{1t_n} - \hat{\theta}_{1t_{n-1}} > 0$  then  $B_{1t_n} = B_{2t_n} \cdot \frac{1+\lambda}{1-\lambda}$ . If  $\hat{\theta}_{1t_n} - \hat{\theta}_{1t_{n-1}} < 0$  then  $B_{1t_n} = B_{2t_n} \cdot \frac{1-\lambda}{1+\lambda}$ .

Theorem 4.2 shows that aside from a stylized special case, any transaction cost equilibrium with a bank account must have non-constant interest rates. The proof is presented in Section 5.

**Theorem 4.2.** *Let  $\tilde{\beta}_i := \beta_i + \alpha_i \mu_i - \frac{\alpha_i^2 \sigma_i^2}{2}$ , and suppose that  $\tilde{\beta}_i$  and  $\lambda$  are strictly positive for  $i = 1, 2$ . Suppose that  $(r_i, \hat{c}_i, \hat{\theta}_i)_{i=1,2}$  is a transaction cost equilibrium with a bank account. If  $r_1, r_2$  are strictly positive constants, then the following must hold:*

- (1) *The agents' parameters satisfy  $\tilde{\beta}_1 = \tilde{\beta}_2$ .*
- (2) *No trading occurs in equilibrium:  $\hat{\theta}_{it_n} - \hat{\theta}_{it_{n-1}} = 0$  for  $i = 1, 2$  and  $n \geq 1$ .*
- (3) *The shadow rates are identical and satisfy*

$$r_1 = r_2 = \frac{1}{\Delta} \left( e^{\tilde{\beta}_1 \Delta} - 1 \right).$$

## 5 Proofs

We begin by proving Theorem 2.2 in the discrete-time case.

*Proof.* We check that  $\hat{c}_i$  is admissible, by noting that

$$\begin{aligned} & \mathbb{E} \left[ \exp \left( -\beta_i t_n - \alpha_i r_i X_{it_n}^{\hat{c}_i} - \alpha_i Y_{it_n} \right) \right] \\ &= \mathbb{E} \left[ \exp \left( -\alpha_i r_i X_{i0} - n \log(1 + r_i \Delta) - \frac{\alpha_i^2 \sigma_i^2}{2} n \Delta - \alpha_i \sum_{k=1}^n \sqrt{\Delta} \sigma_i Z_{it_k} \right) \right] \\ &= (1 + r_i \Delta)^{-n} \exp(-\alpha_i r_i X_{i0}) \longrightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

We have that  $c \mapsto -e^{-\alpha_i c} + e^{-\beta_i \Delta} \mathbb{E} \left[ J \left( x(1 + r_i \Delta) + y - c, y + \mu_i \Delta + \sigma_i \sqrt{\Delta} Z \right) \right]$  is maximized for  $\hat{c} = \hat{c}(x, y) = r_i x + y + \frac{\tilde{\beta}_i}{\alpha_i r_i} - \frac{1}{\alpha_i r_i \Delta} \log(1 + r_i \Delta)$ , where  $Z$  denotes a standard normal random variable. Thus,

$$\left\{ -\sum_{k=0}^{n-1} e^{-\alpha_i c_{t_k}} + e^{-\beta_i t_n} J_i^\Delta(X_{it_n}^c, Y_{it_n}) \right\}_{n \geq 0}$$

is a supermartingale for all  $c \in \mathcal{A}_i^\Delta$  and is a martingale for  $c = \hat{c}_i \in \mathcal{A}_i^\Delta$ .

Therefore, for  $\hat{c}_i$ ,

$$\begin{aligned} J(x, y) &= -\mathbb{E} \left[ \sum_{k=0}^n e^{-\alpha_i \hat{c}_{it_k}} \right] + e^{-\beta_i t_{n+1}} \mathbb{E} \left[ J_i^\Delta \left( X_{it_{n+1}}^{\hat{c}_i}, Y_{it_{n+1}} \right) \right] \\ &= -\mathbb{E} \left[ \sum_{k=0}^{\infty} e^{-\alpha_i \hat{c}_{it_k}} \right] \quad \text{by the transversality condition,} \end{aligned}$$

which implies that  $J_i^\Delta \leq V_i^\Delta$ . Similarly, for any  $c \in \mathcal{A}_i^\Delta$ ,

$$\begin{aligned} J(x, y) &\geq -\mathbb{E} \left[ \sum_{k=0}^n e^{-\alpha_i c_{it_k}} \right] + e^{-\beta_i t_{n+1}} \mathbb{E} \left[ J_i^\Delta \left( X_{it_{n+1}}^c, Y_{it_{n+1}} \right) \right] \\ &= -\mathbb{E} \left[ \sum_{k=0}^{\infty} e^{-\alpha_i c_{it_k}} \right] \quad \text{by the transversality condition.} \end{aligned}$$

Thus,  $J_i^\Delta = V_i^\Delta$ , and  $\hat{c}_i \in \mathcal{A}_i^\Delta$  is the optimal consumption policy. For initial wealth  $x = \theta_{i0} A_i = \theta_{i0}/r_i$ , the optimal wealth policy corresponding to  $\hat{c}_i$  is  $\hat{X}_i = X_i^{\hat{c}_i}$ , and

$$\hat{X}_{it_n} = \frac{\theta_{i0}}{r_i} + \frac{t_n}{\alpha_i r_i} \left( \frac{1}{\Delta} \log(1 + r_i \Delta) - \tilde{\beta}_i \right).$$

□

We now move towards the proof of Theorem 2.4. The self-financing condition (2.1) with the optimal policies (2.2) and (2.3) imply

$$\left( Y_{it_n} - \hat{c}_{it_n} + \hat{\theta}_{it_n} \right) \Delta = \left( \hat{\theta}_{it_n} - \hat{\theta}_{it_{n-1}} \right) A_{it_n} = \frac{1}{\alpha_i r_i} \left( \log(1 + r_i \Delta) - \tilde{\beta}_i \Delta \right). \quad (5.1)$$

For  $i = 1, 2$ , we define

$$F_i(r) := \frac{1}{\alpha_i r} \left( \log(1 + r \Delta) - \tilde{\beta}_i \Delta \right), \quad r > 0. \quad (5.2)$$

Using Definition 2.3 part (i), we seek solutions  $r_1, r_2 > 0$  such that

$$F_1(r_1) + F_2(r_2) = \lambda (|F_1(r_1)| + |F_2(r_2)|).$$

By rewriting this equation and including Condition (iii) from Definition 2.3, we seek  $r_1, r_2 > 0$  such that

$$F_2(r_2) = \begin{cases} -\frac{1-\lambda}{1+\lambda} F_1(r_1), & \text{if } F_1(r_1) \geq 0, \\ -\frac{1+\lambda}{1-\lambda} F_1(r_1), & \text{if } F_1(r_1) \leq 0. \end{cases} \quad (5.3)$$

and

$$\frac{r_2}{r_1} = \begin{cases} \frac{1+\lambda}{1-\lambda}, & \text{if } F_1(r_1) > 0, \\ \frac{1-\lambda}{1+\lambda}, & \text{if } F_1(r_1) < 0, \\ \in \left[ \frac{1-\lambda}{1+\lambda}, \frac{1+\lambda}{1-\lambda} \right], & \text{if } F_1(r_1) = 0, \end{cases} \quad (5.4)$$

**Proposition 5.1.** *Let  $\tilde{\beta}_i := \beta_i + \alpha_i \mu_i - \alpha_i^2 \sigma_i^2 / 2$ , and suppose that  $\tilde{\beta}_i$  is strictly positive for  $i = 1, 2$ . There exists a unique strictly positive solution pair  $r_1, r_2$  to (5.3) and (5.4).*

**Case 1:** *If  $\frac{e^{\tilde{\beta}_1 \Delta} - 1}{e^{\tilde{\beta}_2 \Delta} - 1} \in \left[ \frac{1-\lambda}{1+\lambda}, \frac{1+\lambda}{1-\lambda} \right]$ , then*

$$r_1 = \frac{e^{\tilde{\beta}_1 \Delta} - 1}{\Delta} \quad \text{and} \quad r_2 = \frac{e^{\tilde{\beta}_2 \Delta} - 1}{\Delta}. \quad (5.5)$$

**Case 2:** *If we have  $\frac{e^{\tilde{\beta}_2 \Delta} - 1}{e^{\tilde{\beta}_1 \Delta} - 1} > \frac{1+\lambda}{1-\lambda}$ , then the unique positive solutions satisfy*

$$r_1 \in \left( \frac{e^{\tilde{\beta}_1 \Delta} - 1}{\Delta}, \frac{1-\lambda}{1+\lambda} \left( \frac{e^{\tilde{\beta}_2 \Delta} - 1}{\Delta} \right) \right) \quad \text{and} \quad r_2 = \frac{1+\lambda}{1-\lambda} r_1.$$

*Proof.* We show the existence of the unique solution pair by examining both cases. Suppose that

$$\frac{e^{\tilde{\beta}_1 \Delta} - 1}{e^{\tilde{\beta}_2 \Delta} - 1} \in \left[ \frac{1-\lambda}{1+\lambda}, \frac{1+\lambda}{1-\lambda} \right].$$

Then  $r_1$  and  $r_2$  as in (5.5) is the unique solution to (5.3) and (5.4) such that  $F_1(r_1) = F_2(r_2) = 0$ .

To show uniqueness, we proceed by contradiction. Assume for the sake of contradiction that there exist strictly positive solutions  $r_1, r_2$  such that  $F_1(r_1) > 0$ . We have that  $F_1(r_1) > 0$  if and only if  $F_2(r_2) < 0$ ,  $r_1 \Delta > e^{\tilde{\beta}_1 \Delta} - 1$ , and  $r_2 \Delta < e^{\tilde{\beta}_2 \Delta} - 1$ . Then by (5.4),

$$\frac{e^{\tilde{\beta}_1 \Delta} - 1}{e^{\tilde{\beta}_2 \Delta} - 1} < \frac{r_1}{r_2} = \frac{1-\lambda}{1+\lambda} \leq \frac{e^{\tilde{\beta}_1 \Delta} - 1}{e^{\tilde{\beta}_2 \Delta} - 1},$$

which is a contradiction. Here, we have used that  $\tilde{\beta}_1, \tilde{\beta}_2$  are strictly positive to ensure that  $e^{\tilde{\beta}_i \Delta} - 1 > 0$ . The same argument applies to rule out the case when  $F_1(r_1) < 0$  and  $F_2(r_2) > 0$ . Therefore, we must have that  $F_1(r_1) = F_2(r_2) = 0$ , in which case  $r_1 = \frac{e^{\tilde{\beta}_1 \Delta} - 1}{\Delta}$  and  $r_2 = \frac{e^{\tilde{\beta}_2 \Delta} - 1}{\Delta}$ .

We now consider the existence of a solution in Case 2. For  $F_1(r_1) > 0$ , (5.3) and (5.4) reduce to solving for  $r_1 > 0$  such that

$$\left( 1 + r_1 \Delta \cdot \frac{1+\lambda}{1-\lambda} \right)^{1/\alpha_2} (1 + r_1 \Delta)^{1/\alpha_1} = \exp \left( \left( \frac{\tilde{\beta}_1}{\alpha_1} + \frac{\tilde{\beta}_2}{\alpha_2} \right) \Delta \right), \quad (5.6)$$

while  $r_2 = \frac{1+\lambda}{1-\lambda} \cdot r_1$ . The assumption that  $\tilde{\beta}_1, \tilde{\beta}_2$  are strictly positive ensures that the right hand side of (5.6) is strictly bigger than 1. We note that  $x \mapsto \left(1 + x \cdot \frac{1+\lambda}{1-\lambda}\right)^{1/\alpha_2} (1+x)^{1/\alpha_1}$  strictly increases from 1 to  $\infty$  for  $x \in [0, \infty)$ . Thus, there exists a unique solution  $r_1 > 0$  to (5.6). Moreover,  $\frac{e^{\tilde{\beta}_2 \Delta} - 1}{e^{\tilde{\beta}_1 \Delta} - 1} > \frac{1+\lambda}{1-\lambda}$  implies that

$$r_1 \in \left( \frac{e^{\tilde{\beta}_1 \Delta} - 1}{\Delta}, \frac{1-\lambda}{1+\lambda} \left( \frac{e^{\tilde{\beta}_2 \Delta} - 1}{\Delta} \right) \right).$$

We show uniqueness for Case 2 by contrapositive, which will rule out the possibility of finding solutions for which  $F_1(r_1) \leq 0$ . Suppose that there exist strictly positive solutions  $r_1, r_2$  such that  $F_1(r_1) \leq 0$ . Since  $F_1(r_1) \leq 0$  if and only if  $F_2(r_2) \geq 0$ ,  $r_1 \Delta \leq e^{\tilde{\beta}_1 \Delta} - 1$ , and  $r_2 \Delta \geq e^{\tilde{\beta}_2 \Delta} - 1$ , we have that

$$\frac{1+\lambda}{1-\lambda} \geq \frac{r_2}{r_1} \geq \frac{e^{\tilde{\beta}_2 \Delta} - 1}{e^{\tilde{\beta}_1 \Delta} - 1},$$

as desired.  $\square$

*Proof of Theorem 2.4.* By Theorem 2.2 and Definition 2.3, we must solve (5.3) and (5.4) for the equilibrium shadow interest rates. Proposition 5.1 provides us with the existence and uniqueness of positive shadow interest rates, as desired.

The agents choose not to trade in Case 1, and the market interest rate cannot be uniquely determined. The annuity values  $A$  consistent with this equilibrium must satisfy

$$A \in \left[ \frac{A_1}{1+\lambda}, \frac{A_1}{1-\lambda} \right] \cap \left[ \frac{A_2}{1+\lambda}, \frac{A_2}{1-\lambda} \right] = \left[ \frac{\max(A_1, A_2)}{1+\lambda}, \frac{\min(A_1, A_2)}{1-\lambda} \right].$$

Since  $A_i = \frac{\Delta}{e^{\tilde{\beta}_i \Delta} - 1}$  for  $i = 1, 2$ , we can rewrite the above interval as

$$A \in \left[ \frac{\Delta}{(1+\lambda)(e^{\min(\tilde{\beta}_1, \tilde{\beta}_2) \Delta} - 1)}, \frac{\Delta}{(1-\lambda)(e^{\max(\tilde{\beta}_1, \tilde{\beta}_2) \Delta} - 1)} \right].$$

This interval is nonempty by (2.5). Since  $A = 1/r$ , we have that

$$r \in \left[ \frac{1-\lambda}{\Delta} (e^{\max(\tilde{\beta}_1, \tilde{\beta}_2) \Delta} - 1), \frac{1+\lambda}{\Delta} (e^{\min(\tilde{\beta}_1, \tilde{\beta}_2) \Delta} - 1) \right] \neq \emptyset.$$

Trading occurs in Case 2, in which case we are able to determine a unique market interest rate. When

$$\frac{e^{\tilde{\beta}_2 \Delta} - 1}{e^{\tilde{\beta}_1 \Delta} - 1} > \frac{1+\lambda}{1-\lambda},$$



we have that  $r_2 = \frac{1+\lambda}{1-\lambda} \cdot r_1$  while  $r_1 > 0$  solves (5.6). In this case,  $1/r_1 = A_1 = A(1+\lambda) = (1+\lambda)/r$ , which implies that  $r = (1+\lambda)r_1$ . Similarly,  $r = (1-\lambda)r_2$ . Therefore, the market interest rate  $r > 0$  is determined by

$$\left(1 + \frac{r\Delta}{1-\lambda}\right)^{\frac{1}{\alpha_2\Delta}} \left(1 + \frac{r\Delta}{1+\lambda}\right)^{\frac{1}{\alpha_1\Delta}} = e^{\left(\frac{\tilde{\beta}_1}{\alpha_1} + \frac{\tilde{\beta}_2}{\alpha_2}\right)},$$

and the shadow interest rates are given in terms of

$$r = r_1(1+\lambda) = r_2(1-\lambda).$$

□

In continuous time, the proofs of Theorems 3.2 and 3.4 mirror their discrete-time counterparts. The continuous-time analog of  $F_i$  defined in (5.2) is given for  $i = 1, 2$  by

$$F_i(r) := \frac{1}{\alpha_i} \left(1 - \frac{\tilde{\beta}_i}{r}\right), \quad r > 0.$$

We now prove Theorem 3.6.

*Proof of Theorem 3.6.* Since  $\tilde{\beta}_2/\tilde{\beta}_1 > \frac{1+\lambda}{1-\lambda}$ , Theorem 3.4 shows that trade occurs for  $\Delta = 0$  and  $r(0)$  is given uniquely by (3.6). Moreover,

$$\frac{e^{\tilde{\beta}_2\Delta} - 1}{e^{\tilde{\beta}_1\Delta} - 1} \longrightarrow \frac{\tilde{\beta}_2}{\tilde{\beta}_1} \quad \text{as } \Delta \rightarrow 0,$$

which by Theorem 2.4 implies that trade occurs for sufficiently small  $\Delta > 0$ . In this case,  $r(\Delta) > 0$  is given uniquely by the solution to (2.7).

For  $(\Delta, r) \in [0, \infty) \times (0, \infty)$ , we define

$$G(\Delta, r) := \begin{cases} \left(1 + \frac{r\Delta}{1+\lambda}\right)^{\frac{1}{\alpha_1\Delta}} \left(1 + \frac{r\Delta}{1-\lambda}\right)^{\frac{1}{\alpha_2\Delta}}, & \text{for } \Delta > 0 \\ \exp\left(r\left(\frac{1}{\alpha_1(1+\lambda)} + \frac{1}{\alpha_2(1-\lambda)}\right)\right), & \text{for } \Delta = 0. \end{cases}$$

For sufficiently small  $\Delta > 0$  and  $\Delta = 0$ ,  $r(\Delta)$  is chosen such that  $G(\Delta, r(\Delta)) = \exp\left(\frac{\tilde{\beta}_1}{\alpha_1} + \frac{\tilde{\beta}_2}{\alpha_2}\right)$ . Since  $G$  is smooth on  $[0, \infty) \times (0, \infty)$  and  $\frac{\partial G}{\partial r}(0, r(0)) \neq 0$  (a one-sided derivative), the implicit function theorem implies that  $r(\Delta) \rightarrow r(0)$  as  $\Delta \rightarrow 0$ . □

Finally, we show Theorem 4.2.

*Proof of Theorem 4.2.* Assume that  $r_1, r_2$  are strictly positive constants. By modifying (5.1) to account for a traded bank account, we arrive at the same form of  $F_i$  as in (5.2).

By Definition 4.1 (iii),  $r_1$  and  $r_2$  must satisfy (5.3) and

$$\left(\frac{1+r_1}{1+r_2}\right)^n \in \left[\frac{1-\lambda}{1+\lambda}, \frac{1+\lambda}{1-\lambda}\right], \quad \text{for } n \geq 0,$$

while for each  $n \geq 0$ ,  $B_{1t_n} = B_{2t_n} \cdot \frac{1+\lambda}{1-\lambda}$  if  $\hat{\theta}_{1t_n} - \hat{\theta}_{1t_{n-1}} > 0$ , and  $B_{1t_n} = B_{2t_n} \cdot \frac{1-\lambda}{1+\lambda}$  if  $\hat{\theta}_{1t_n} - \hat{\theta}_{1t_{n-1}} < 0$ . Since  $\lambda \neq 0$ , we must have that  $r_1 = r_2$  and  $\hat{\theta}_{1t_n} - \hat{\theta}_{1t_{n-1}} = \hat{\theta}_{2t_n} - \hat{\theta}_{2t_{n-1}} = 0$  for all  $n \geq 1$ . Moreover, (5.3) implies that  $F_1(r_1) = F_2(r_2) = 0$ , and thus

$$\log(1 + r_1 \Delta) = \tilde{\beta}_1 \Delta = \tilde{\beta}_2 \Delta = \log(1 + r_2 \Delta),$$

as desired. □

## References

- [1] Adrian Buss, Raman Uppal, and Grigory Vilkov. Asset prices in general equilibrium with recursive utility and illiquidity induced by transactions costs. Working paper, January 2014.
- [2] Laurent E. Calvet. Incomplete markets and volatility. *Journal of Economic Theory*, 98(2):295–338, June 2001.
- [3] Peter Ove Christensen, Kasper Larsen, and Claus Munk. Equilibrium in securities markets with heterogeneous investors and unspanned income risk. *Journal of Economic Theory*, 147(3):1035–1063, May 2012.
- [4] Jakša Cvitanić and Ioannis Karatzas. Hedging and portfolio optimization under transaction costs: A martingale approach. *Mathematical Finance*, 6(2):133–165, April 1996.
- [5] Christoph Czichowsky and Walter Schachermayer. Duality theory for portfolio optimisation under transaction costs. *The Annals of Applied Probability*, 26(3):1888–1941, 2016.
- [6] Eduardo Dávila. Optimal financial transaction taxes. Working paper, December 2016.
- [7] M. H. A. Davis and A. R. Norman. Portfolio selection with transaction costs. *Mathematics of Operations Research*, 15(4):676–713, November 1990.
- [8] Ming Huang. Liquidity shocks and equilibrium liquidity premia. *Journal of Economic Theory*, 109(1):104–129, March 2003.
- [9] Karel Janecek and Steven E. Shreve. Asymptotic analysis for optimal investment and consumption with transaction costs. *Finance and Stochastics*, 8(2):181–206, May 2004.

- [10] Elyès Jouini and Hédi Kallal. Martingales and arbitrage in securities markets with transaction costs. *Journal of Economic Theory*, 66(1):178–197, June 1995.
- [11] Jan Kallsen and Johannes Muhle-Karbe. On using shadow prices in portfolio optimization with transaction costs. *The Annals of Applied Probability*, 20(4):1341–1358, August 2010.
- [12] Jan Kallsen and Johannes Muhle-Karbe. The general structure of optimal investment and consumption with small transaction costs. *Mathematical Finance*, September 2015.
- [13] Kasper Larsen and Tanawit Sae-Sue. Radner equilibrium in incomplete Lévy models. *Mathematics and Financial Economics*, 10(3):321–337, January 2016.
- [14] Andrew W. Lo, Harry Mamaysky, and Jiang Wang. Asset prices and trading volume under fixed transactions costs. *Journal of Political Economy*, 112(5), 2004.
- [15] Michael J.P Magill and George M Constantinides. Portfolio selection with transactions costs. *Journal of Economic Theory*, 13(2):245–263, October 1976.
- [16] S. E. Shreve and H. M. Soner. Optimal investment and consumption with transaction costs. *The Annals of Applied Probability*, 4(3):609–692, August 1994.
- [17] H. Mete Soner and Nizar Touzi. Homogenization and asymptotics for small transaction costs. *SIAM Journal on Control and Optimization*, 51(4):2893–2921, January 2013.
- [18] Dimitri Vayanos. Transaction costs and asset prices: A dynamic equilibrium model. *Review of Financial Studies*, 11(1):1–58, January 1998.
- [19] Dimitri Vayanos and Jean-Luc Vila. Equilibrium interest rate and liquidity premium with transaction costs. *Economic Theory*, 13(3), 1999.
- [20] Neng Wang. Caballero meets Bewley: The permanent-income hypothesis in general equilibrium. *American Economic Review*, 93:927–936, 2003.